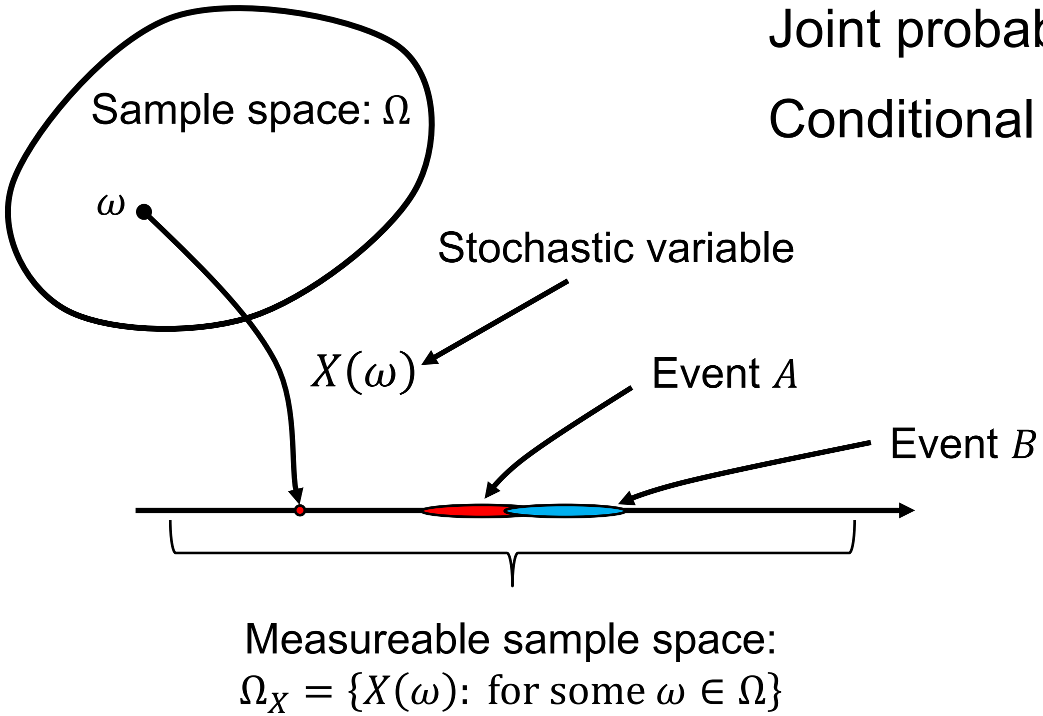
TSKS01 DIGITAL COMMUNICATION

Repetition and Examples

STOCHASTIC VARIABLES

Probability, Stochastic Variable, and Events



Total probability: $\Pr\{\Omega_X\} = 1$

Probability of event A: $\Pr\{A\} \in [0,1]$

Joint probability: $\Pr\{A, B\}$

Conditional probability: $\Pr\{A|B\} = \frac{\Pr\{A,B\}}{\Pr\{B\}}$

Probabilities and Distributions

Probability distribution function:

$$F_X(x) = \Pr\{X \le x\} \in [0,1]$$

Probability density function (PDF):

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Properties:

$F_X(x)$ is non-decreasing,

$F_X(x) \ge 0$ and $f_X(x) \ge 0$ for all x,

$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$,

$\Pr\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(x)dx.$

Example: Stochastic variables

Game based on tossing two fair coins:

Two heads	+400
Two tails	–100
One tail, one head	–200

$$\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$$

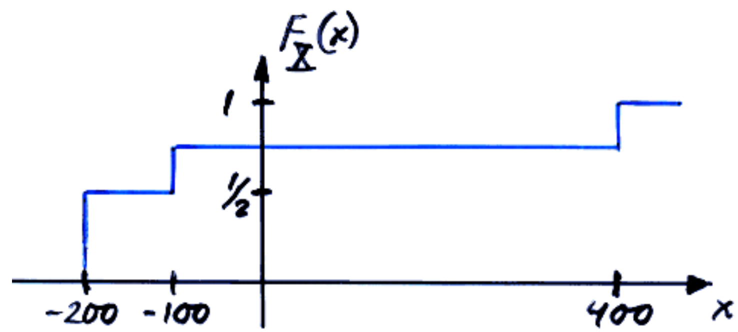
$$\Omega_X = \{-200, -100, +400\}$$

$$X(\omega) = \begin{cases} +400, & \omega = HH, \\ -100, & \omega = TT, \\ -200, & \omega \in \{HT, TH\} \end{cases}$$

Determine distribution and density:

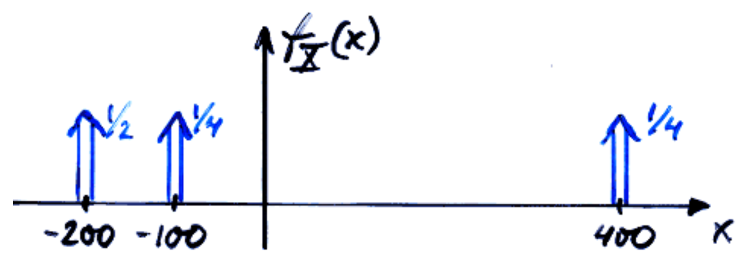
$$F_X(x) = \begin{cases} 0, & x < -200 \\ 0.5, & -200 \leq x < -100 \\ 0.75, & -100 \leq x < 400 \\ 1, & x \geq 400 \end{cases}$$

$$= \frac{1}{2}u(x+200) + \frac{1}{4}u(x+100) + \frac{1}{4}u(x-400)$$



$$f_X(x) = \frac{d}{dx}F_X(x)$$

$$= \frac{1}{2}\delta(x+200) + \frac{1}{4}\delta(x+100) + \frac{1}{4}\delta(x-400)$$



Expectation and Variance

Expectation (mean):

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx$$

Quadratic mean (power):

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

Variance:

$$\begin{aligned} Var\{X\} &= E\{(X - E\{X\})^2\} \\ &= E\{X^2\} - (E\{X\})^2 \end{aligned}$$

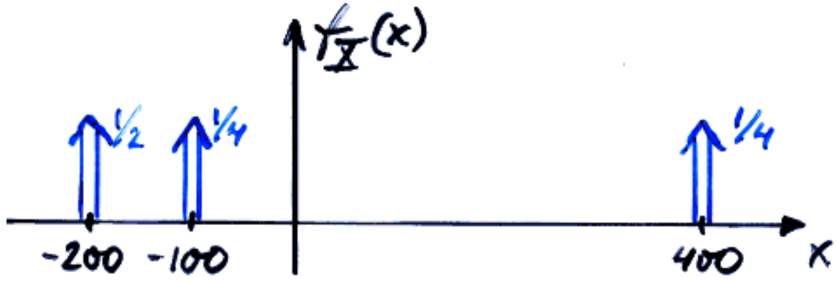
Common notation:

$m_X = E\{X\}$,
$m_Y = E\{Y\}$

$\sigma_X^2 = \mathrm{Var}\{X\}$

σ_X is called the standard deviation

Example (cont.): Stochastic variables



$$f_X(x) = \frac{1}{2}\delta(x+200) + \frac{1}{4}\delta(x+100) + \frac{1}{4}\delta(x-400)$$

Determine mean and variance:

$$E\{X\} = \int_{-\infty}^{\infty} x\left(\frac{1}{2}\delta(x+200) + \frac{1}{4}\delta(x+100) + \frac{1}{4}\delta(x-400)\right)dx$$

$$= -200\cdot\frac{1}{2} - 100\cdot\frac{1}{4} + 400\cdot\frac{1}{4} = -25$$

$$E\{X^2\} = (-200)^2\cdot\frac{1}{2} + (-100)^2\cdot\frac{1}{4} + (400)^2\cdot\frac{1}{4} = 62500$$

$$\mathrm{Var}\{X\} = E\{X^2\} - (E\{X\})^2 = 61875$$

Example: Mean and variance of distribution

Uniform distribution:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x < b \\ 0, & \text{elsewhere} \end{cases}$$

Mean:

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$$

Variance:

$$Var\{X\} = E\{X^2\} - (E\{X\})^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

Multi-Dimensional Stochastic Variables

Distribution: $F_{X_1,\dots,X_N}(x_1,\dots,x_N) = \Pr\{X_1 \le x_1,\dots,X_N \le x_N\}$

Density: $f_{X_1,\dots,X_N}(x_1,\dots,x_N) = \frac{\partial^N}{\partial x_1 \cdots \partial x_N} F_{X_1,\dots,X_N}(x_1,\dots,x_N)$

Vector notation: $\bar{X} = (X_1,\dots,X_N)$, $\bar{x} = (x_1,\dots,x_N)$, $F_{\bar{X}}(\bar{x})$, $f_{\bar{X}}(\bar{x})$

Mutual independence:

Marginal distribution and density

$$F_{\bar{X}}(\bar{x}) = \prod_{i=1}^{N} F_{X_i}(x_i) \qquad f_{\bar{X}}(\bar{x}) = \prod_{i=1}^{N} f_{X_i}(x_i)$$

Covariance (pairwise): $\mathrm{Cov}\{X_i, X_j\} = E\{X_i X_j\} - E\{X_i\}E\{X_j\}$

Pairwise uncorrelated if X_i and X_j if $\mathrm{Cov}\{X_i, X_j\} = 0$ for all i, j

Example: Uncorrelated ≠ Independent

The stochastic variable X is $+1$, 0, and -1 with equal probability

Mean value: $m_X = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) = 0$

Consider the stochastic variable $Y = X^2$. Are X and Y correlated?

$$
\begin{aligned}
Cov\{X, Y\} &= E\{XY\} - m_X m_Y \\
&= E\{X^3\} - 0 \\
&= \frac{1}{3} \cdot 1^3 + \frac{1}{3} \cdot 0^3 + \frac{1}{3} \cdot (-1)^3 = 0
\end{aligned}
$$

Uncorrelated!

Are X and Y independent? No, since $Y = X^2$